Ray tracing equations in transversely isotropic media

Cosmin Macesanu and Faruq Akbar, Seimax Technologies, Inc.

SUMMARY

We discuss a simple, compact approach to deriving ray tracing equations in transversely isotropic media. The general equations derived here are given in terms of the ray slowness vector and the direction of the axis of symmetry of the medium, which can change orientation depending on location. The standard Thomsen parameters are used instead of stress-energy tensor components, thus allowing for easy connection with experimentally relevant quantities. The TTI results can easily be simplified to the VTI and HTI cases (corresponding to particular orientations of the symmetry axis). Both weak and strong anisotropy are considered. The accuracy of our formulas is verified by comparing traveltimes with 2-way wave equation numerical results.

INTRODUCTION

Ray tracing is an essential tool for many modern imaging algorithms (Kirchhoff and Beam migration) and velocity update methods (tomography). Ray tracing equations in isotropic media are well understood and documented. However, taking into account anisotropic effects (due to cracks and layers of various strata in the geological model) has become quite important for modern seismological applications. In this paper we present a compact system of equations applicable for anisotropic media with transverse symmetry (which, depending on the direction of the symmetry axis, goes under the name of VTI for vertical, HTI for horizontal, or TTI for tilted transverse isotropy).

The general theory of ray tracing in heterogenous anisotropic media is discussed in Cerveny (2001). Thomsen (1986) defines anisotropic parameters relevant to practical use and derives results for phase and group velocity for transversely isotropic media. Sena (1991) (see also Dong et al. (2000)) present explicit results for velocity and traveltimes in HTI media, with azimuth-oriented axis of symmetry. The standard ray tracing equations for a medium with no particular symmetry involve evaluating up to 81 terms in a single equation, which is quite expensive numerically. Symmetries in the medium, or using the weak anisotropy approximation, will reduce the complexity of the equations involved and speed up the computational process. Psencik and Farra (2005) (also see Dehghan et al. (2005)) introduce first order ray-tracing (FORT) equations for a general anisotropic medium, applicable for weak anisotropy values. They also present explicit FORT equations for Orthorhombic, HTI oriented along the X axis, and VTI media.

Here we extend these results for transversely tilted isotropic (TTI) media to large values of anisotropic parameters. We derive a relatively simple and elegant system of equations that accommodates varying velocity, anisotropic parameters, and changes in the orientation of the symmetry axis. We also give the simplified results applicable for VTI and HTI media with weak anisotropy.

RAY TRACING EQUATIONS

Following Cerveny (2001), the ray tracing equations can be written in terms of the eigenvalues $G(p,x)$ of the Christoffel matrix Γ :

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{1}{2}\frac{\partial G(p,x)}{\partial p_i} \; ; \; \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{1}{2}\frac{\partial G(p,x)}{\partial x_i} \; . \qquad (1)$$

Here x_i are the coordinates of a point on the ray, and p_i are the components of the slowness vector ($p_i = n_i/v$) along the ray at that point. In the above equations, x_i and p_i are treated as independent variables.

The Christoffel matrix components are given in terms of the elastic stress tensor

$$\Gamma_{ij}(x,p) = c_{ijkl}(x)p_j p_l \; ,$$

and the functions G are solutions of the eigenvalue equation

$$\det(\Gamma_{ij} - G(x,p)\delta_{ij}) \; = \; 0 \; . \qquad (2)$$

For a transversely isotropic medium, the non-zero components of the stress tensor in a coordinate system with the symmetry axis in the z direction are (following the Voight notation and Thomsen (1986)):

$$c_{1111} = c_{2222} = C_{11} \; , \; c_{3333} = C_{33} \; ,$$
$$c_{1122} = C_{11} - 2C_{66} \; , \; c_{1133} = c_{2233} = C_{13}$$
$$c_{2323} = c_{1313} = C_{44} \; , \; c_{1212} = C_{66} \qquad (3)$$

In terms of the standard weak anisotropy Thomsen parameters:

$$C_{33} = \rho v_o^2 \; , \; C_{11} = \rho v_o^2(1+2\varepsilon) \; , \; C_{13} = \rho v_o^2\sqrt{1+2\delta} \; , \qquad (4)$$

where v_o is the p-wave velocity along the symmetry axis. In the pseudoacoustic approximation we can set $C_{44} = C_{66} = 0$.

The eigenvalue equation (2) has three solutions for $G(p)$, the largest one corresponding to the (pseudo) p-wave propagation, and the other two corresponding to shear waves modes. We will discuss the p-wave propagation in the following, since the treatment of shear waves is quite similar.

VTI in the weak anisotropy limit

In the limit of weak anisotropy, the eigenvalue equation for the p-wave then becomes:

$$G(p_i,x) = v_o^2\left[p^2 + 2\delta(x)\frac{p_z^2 p_r^2}{p^2} + 2\varepsilon(x)\frac{p_r^4}{p^2}\right] \; , \qquad (5)$$

with $p_r^2 = p_x^2 + p_y^2$ the radial component of the slowness. In terms of the angle θ between the vertical (symmetry) axis and the slowness vector ($n_z = \cos\theta$):

$$\begin{aligned} G(p_i,x) &= v_o^2 p^2\left(1+2\delta\cos^2\theta\sin^2\theta + 2\varepsilon\sin^4\theta\right) \\ &= v_o^2 p^2\left(1-2\eta\cos^2\theta\sin^2\theta + 2\varepsilon\sin^2\theta\right) \end{aligned} \qquad (6)$$

with $\eta = \varepsilon - \delta$. The equation $G(p,x) = 1$ will give us the magnitude of the phase velocity v as a function of the VTI medium parameters v_0, ε, δ, and the angle θ.

To compute the change in coordinates along the ray (left hand equation in 1) we have to take the derivation of G with respect to p_i variables, while the x variables are kept constant (that is, the $v_0(x), \varepsilon(x)$ and $\delta(x)$ functions will be kept constant). Note that the eigenvalues G are second order homogenous functions in the p variables; this means that they can be written in the generic form:

$$G(p_i,x) = v_0^2 p^2 g(\sin^2\theta, x) \ ;$$

therefore,

$$\frac{1}{2v_0^2}\frac{\partial G(p,x)}{\partial p_i} = p_i g(\theta,x) + \frac{p^2}{2}\frac{\mathrm{d}g(\theta,x)}{\mathrm{d}\sin^2\theta}\frac{\partial \sin^2\theta}{\partial p_i} \ .$$

With $\sin^2\theta = (p_x^2 + p_y^2)/p^2$:

$$\frac{p^2}{2}\frac{\partial \sin^2\theta}{\partial p_z} = -p_z \sin^2\theta \ ; \ \frac{p^2}{2}\frac{\partial \sin^2\theta}{\partial p_{x,y}} = p_{x,y}\cos^2\theta \ . \quad (7)$$

The equations for ray coordinates change in weakly anisotropic VTI media become :

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= p_z v_0^2 (1 - 2\eta \sin^4\theta) \\ \frac{\mathrm{d}(x,y)}{\mathrm{d}t} &= p_{x,y} v_0^2 (1 + 2\varepsilon - 2\eta\cos^4\theta) \ . \end{aligned} \quad (8)$$

Conversely, to compute the change in slowness vector along the ray, we keep the p variables (therefore θ) constant in the right hand equation (1). Then, we have

$$\frac{-1}{p^2}\frac{\mathrm{d}p_i}{\mathrm{d}t} = v_o \nabla v_0 \ g(\theta) + v_o^2 \sin^2\theta \left(\nabla\varepsilon - \nabla\eta\cos^2\theta\right) \ , \quad (9)$$

or, if instead of anisotropic parameter ε one uses the horizontal velocity $v_h(x) = v_0(x)\sqrt{1+2\varepsilon(x)}$:

$$\begin{aligned} -\frac{1}{p^2}\frac{dp_i}{dt} &= v_o \nabla v_o \cos^2\theta (1 - 2\eta\sin^2\theta) - \\ & \quad v_o^2 \nabla\eta \sin^2\theta\cos^2\theta + v_h \nabla v_h \sin^2\theta \ . \end{aligned} \quad (10)$$

TTI in the weak anisotropy limit

Transversely tilted anisotropy is esentially VTI along an arbitrary symmetry axis, whose orientation may vary in space. Let us assume the direction of the new symmetry axis is given by a new vector field $k(x)$ (of unit magnitude $k^2 = 1$). One can then straightforwardly generalize the eigenvalue equation (5) by replacing the the slowness component p_z with the projection on the new symmetry axis pk:

$$\begin{aligned} G(p_i,x) &= v_o^2 p^2 \left[1 + 2\delta(x)\frac{(pk)^2(p^2-(pk)^2)}{p^4} \right. \\ & \quad \left. + 2\varepsilon(x)\frac{(p^2-(pk)^2)^2}{p^4} \right] \ , \end{aligned} \quad (11)$$

(see, for example, Eq. 24 in Dehghan et al. (2005)). Expressed in terms of the angle between the slowness vector and the symmetry axis $\cos\theta = pk/p$, equations 6 stay unchanged, the difference in the TTI case being that the angle θ is dependent on the spatial location x.

To compute the change in coordinates along the ray, the equivalent of Eqs. (7) are

$$\begin{aligned} \frac{p^2}{2}\frac{\partial \sin^2\theta}{\partial p_i} &= \frac{-p^2}{2}\frac{\partial\cos^2\theta}{\partial p_i} = \frac{-p^2}{2}\frac{\partial}{\partial p_i}\frac{(pk)^2}{p^2} \\ = -pk\left(k_i - p_i\frac{pk}{p^2}\right) &= -p\cos\theta\left(k_i - \cos\theta\frac{p_i}{p}\right) \ . \end{aligned} \quad (12)$$

This result reduces to Eqs. (7) if we set $k = (0,0,1)$ for the case of VTI anisotropy.

By evaluating the derivatives, one then obtains:

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}t} &= v_0^2 \left[p_i - 2\eta(pk)\left(k_i - 2k_i\frac{(pk)^2}{p^2} + p_i\frac{(pk)^3}{p^4}\right) \right. \\ & \quad \left. + 2\varepsilon(p_i - k_i(pk)) \right] \ . \end{aligned} \quad (13)$$

The change in slowness vector along the ray is obtained by adding to Eq. (9) terms due to changes in the orientation of the symmetry axis:

$$\begin{aligned} \frac{-1}{p^2}\frac{\mathrm{d}p_i}{\mathrm{d}t} &= v_o \nabla v_0 \ g(\theta) + v_o^2 \sin^2\theta \left(\nabla\varepsilon - \nabla\eta\cos^2\theta\right) \\ & \quad + v_0^2 \left(\varepsilon - \eta\cos(2\theta)\right)\nabla(\sin^2\theta) \ , \end{aligned} \quad (14)$$

with

$$\frac{\partial\sin^2\theta}{\partial x_i} = -\frac{\partial}{\partial x_i}\frac{(pk)^2}{p^2} = -2p_j\frac{pk}{p^2}\frac{\partial k_j}{\partial x_i} \ . \quad (15)$$

Note however that usually the axis of symmetry changes direction rather slowly, so the derivatives $\partial k_j/\partial x_i$ can be expected to be very small numerically. Therefore for practical purposes one can usually neglect the second line terms in Eq. (14) and use Eqs. (9, 10) for the TTI case too. Note that these terms are also typically neglected in efficient numerical implementations of the two-way wave equations for TTI media (see, for example Macesanu (2011)).

HTI anisotropy

For reference, we present formulas specific to the case of HTI anisotropy (where the symmetry axis lies in the xy plane). The ray equations can then be derived from the general TTI case by setting the vector $k = (k_x, k_y, 0) = (\cos\phi, \sin\phi, 0)$, with the angle ϕ constant. One then obtains

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= v_0^2 p_z \left(1 + 2\varepsilon - 2\eta\cos^4\theta\right) \\ \frac{\mathrm{d}(x,y)}{\mathrm{d}t} &= v_0^2 \left[p_{x,y} - 2\eta\cos\theta(pk_{x,y}\cos 2\theta + \right. \\ & \quad \left. p_{x,y}\cos^3\theta) + 2\varepsilon(p_{x,y} - pk_{x,y}\cos\theta) \right] . \end{aligned} \quad (16)$$

For the components of p, Eqs. (9, 10) still apply (where θ is the angle between the slowness vector p and the HTI symmetry axis, and v_h should be interpreted as the vertical velocity rather than horizontal one).

STRONG ANISOTROPY EQUATIONS

For values of the anisotropy parameters greater than some threshold (typically about 0.2), the weak anisotropy approximation may not be accurate enough. In this case, one can use the exact result for the p-wave eigenvalue of the Christoffel matrix. Following the notations (Eqs. 10) in Thomsen (1986), one can write the result as

$$G(\theta,x) = v_0^2 p^2 g(\theta) = v_0^2 p^2 \left(1+\varepsilon \sin^2\theta + D^*(\theta)\right) \;, \quad (17)$$

with

$$D^*(\theta) = \frac{1}{2}\left[\left(\left(1-\frac{v_s^2}{v_o^2}\right)^2 + 4\delta^* \sin^2\theta\cos^2\theta + 4\varepsilon\left(1-\frac{v_s^2}{v_o^2}+\varepsilon\right)\sin^4\theta\right)^{1/2} - 1\right] \quad (18)$$

with $v_s = \sqrt{C_{44}/\rho}$ the shear wave velocity in the direction of the symmetry axis (for both shear modes), and

$$\delta^* = \left(\frac{v_s^2}{v_o^2} + \sqrt{1+2\delta}\right)^2 + \left(1-\frac{v_s^2}{v_o^2}\right)\left(1-\frac{v_s^2}{v_o^2}+\varepsilon\right)$$

(here we keep the definition $\sqrt{1+2\delta} = C_{13}/C_{33}$ used in the weak anisotropy case).

The ray equations can be derived straightforwardly :

$$\frac{1}{v_0^2}\frac{\mathrm{d}x_i}{\mathrm{d}t} = p_i g(\theta) + \frac{p^2}{2}\frac{\partial \sin^2\theta}{\partial p_i}\left(\varepsilon + \frac{\partial D^*(\theta)}{\partial \sin^2\theta}\right) \quad (19)$$

where the derivatives of the $\sin^2\theta$ quantity with respect to the slowness components are given in Eq. (7) for the VTI case and in Eq. (12) for HTI or the general TTI case. Also

$$\frac{-1}{p^2}\frac{\mathrm{d}p_i}{\mathrm{d}t} = v_o \nabla v_0 \, g(\theta) + \frac{v_o^2}{2}\left(\sin^2\theta\nabla\varepsilon + \frac{\partial D^*(\theta)}{\partial x_i}\right) + \frac{v_0^2}{2}\left(\varepsilon + \frac{\partial D^*(\theta)}{\partial \sin^2\theta}\right)\nabla(\sin^2\theta) \quad (20)$$

where again the second line terms can usually be neglected for the TTI case (and are zero for the VTI and HTI cases).

To write down the derivatives of the D^* function, it is convenient to introduce the notations :

$$\rho_s = 1 - \frac{v_s^2}{v_o^2} \;;\; D^*(\theta) = \frac{1}{2}\left(\sqrt{F(\theta)} - 1\right) \;.$$

Then

$$\frac{\partial D^*(\theta)}{\partial \sin^2\theta} = \frac{1}{\sqrt{F(\theta)}}\left[\delta^*\cos(2\theta) + 2\varepsilon(\rho_s+\varepsilon)\sin^2\theta\right]$$

$$\frac{\partial D^*(\theta)}{\partial x_i} = \frac{1}{\sqrt{F(\theta)}}\left[\left(\frac{\rho_s}{2} + \varepsilon\sin^4\theta\right)\nabla\rho_s + \sin^2\theta\cos^2\theta\nabla\delta^* + \sin^4\theta(\rho_s+2\varepsilon)\nabla\varepsilon\right] \;. \quad (21)$$

The above results are applicable for the tracing of rays associated with pseudo-pressure waves. For shear waves, one can use the other two eigenvalue functions:

$$G_{SV}(\theta,x) = v_0^2 p^2\left((v_s/v_0)^2 + \varepsilon\sin^2\theta - D^*(\theta)\right)$$
$$G_{HV}(\theta,x) = v_s^2 p^2\left(1+2\gamma\sin^2\theta\right). \quad (22)$$

NUMERICAL RESULTS

To verify the accuracy (and correctness) of our equations, we run some numerical tests in models with moderate anisotropy. The first test is run in an homogenous VTI model, with a p-velocity $v_0 = 3000m/s, v_s = 0$, and the anisotropic parameters $\varepsilon = 0.2$ and $\delta = 0.1$. We perform ray tracing starting from a surface point, and calculate the traveltimes from the shot location to points on a vertical line situated 3Km from the shot. To evaluate the accuracy of the equations, we compare the result with traveltimes evaluated using a 2-way wave equation for anisotropic media (Duveneck et al. (2008)). Note that the wave equation results are computed in the pseudo-acousic approximation ($v_s = 0$), however, the resulting traveltimes are exact up to numerical errors (that is, there is no approximation related to the magnitude of the anisotropic parameters in the equations being used).

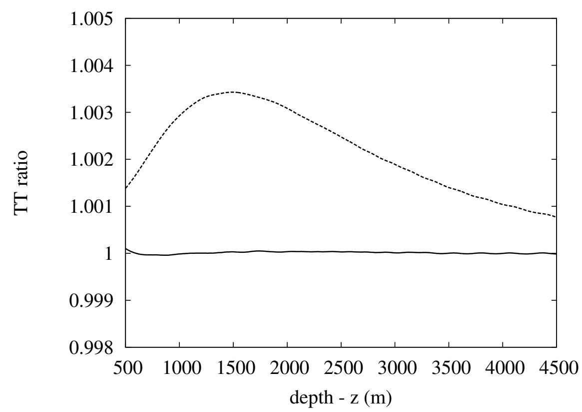

Figure 1: Traveltime ratios for strong (solid line) and weak anisotropy approximation (dashed line) in a homogenous VTI model.

In Fig. 1 we plot the ratio $R_{ray/wave}$ of the traveltimes obtained from ray-tracing versus the traveltimes obtained from wave equation modeling. The solid line corresponds to rays traced with the exact equations (19, 20); note that in this case the two traveltimes agree with an error margin $< 0.01\%$. The dashed line corresponds to the case where the rays are traced using the weak anisotropy approximation (Eqs. 8, 9); the difference from unit in this ratio is due to approximation being used. We see from this plot that the error due to the use of the weak approximation in ray-tracing equations can be quite significant in this case (since relative errors of order 0.1% in traveltime translate into absolute errors of order miliseconds for the times associated with seismic events). These errors seem roughly consistent with those found by Dehghan et al. (2005), which report a weak approximation error of about 0.08% for values of the anisotropy parameters equal to half of the ones used here.

One also notes from Fig. 1 that the greatest errors associated with the weak anisotropy approximation appear at angles close to 60 deg (the relation between depth z in Fig 1. and angle is $\tan\theta' = 3000/z$ – note that this is the ray angle rather than the phase angle θ employed in the previous section). The reason for this behaviour is that anisotropic effects generally increase with angles (indeed, at $\theta = 0$ – along the symmetry axis – the propagation is isotropic). However, at right angle with respect to the symmetry axis the weak approximation becomes exact; one can check this by looking at the weak approximation for the function D^* in Eq. (18):

$$D^*_{weak}(\theta) \;=\; -2\eta \sin^2\theta \cos^2\theta + \varepsilon \sin^4\theta \rightarrow D^*(\theta) \;,$$

in the limit where $\theta \rightarrow 90\,\mathrm{deg}$.

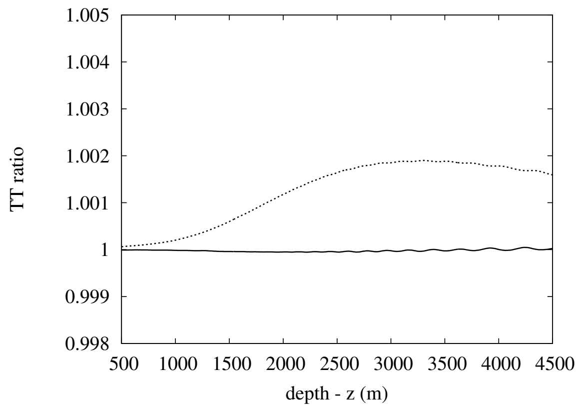

Figure 2: Traveltime ratios for strong (solid line) and weak anisotropy approximation (dashed line) in a v(z) model.

The second model used for testing is a $v(z)$ model, where all parameters vary with depth only. We take $v_0(z) = 2 + 0.5z$ Km/s, $\varepsilon = 0.1z, \delta = 0.05z$ (with depth measured in Km), so that the average of the parameters over the first 4 Km of depth are the same as in the first test. The ratios of the ray-tracing traveltimes and wave equation traveltimes for this model are plotted in Fig. 2 (for the weak approximation and the exact formulas). Again, we note good agreement between the ray-tracing traveltimes and the wave equation ones when using the exact equations (19, 20), and errors of order 0.1% for the weak approximation case. Note that compared with the previous test, the errors in this model are smaller shallow (since the magnitude of the anisotropic parameters is smaller), and somewhat larger at depths greater than 4 Km.

CONCLUSIONS

We have describeded here a compact system of equations for ray-tracing in a transversely isotropic medium. These equations improve the efficiency of numerical implementation of ray-tracing algorithms, compared to methods where the elements of the stress-energy tensor are evaluated individually. We give simpler forms of the equations suitable for weak anisotropy approximation in VTI and HTI media. However, the general results we present are valid for arbitrarily strong anisotropy, and extend previous published work valid in the weak anisotropy limit.

We perform numerical comparison between traveltimes evaluated by ray-tracing, and by using a 2-way wave equation in VTI media. We find very good agreement between the two sets of traveltimes when the full (exact) equations are used for ray tracing. When the weak approximation is used, we find errors of order 0.1% in traveltime evaluation for anisotropic parameters of order 10%. This indicates that using the exact equations is reccommended if one desires good accuracy in models with mild anisotropy.

REFERENCES

Cerveny, V., 2001, Seismic ray theory: Cambridge University Press. (Cambridge Books Online).

Dehghan, K., V. Farra, and L. Nicoletis, 2005, Approximate ray tracing for qp waves in inhomogeneous layered media with weak structural anisotropy: GEOPHYSICS, **72**, SM47–SM60.

Dong, Y., H. Yang, and W. Sun, 2000, An approximation of the group velocity for hti media: SEG Technical Program Expanded Abstracts, 2389–2392.

Duveneck, E., P. Milcik, P. M. Bakker, and C. Perkins, 2008, Acoustic vti wave equations and their application for anisotropic reverse-time migration: SEG Technical Program Expanded Abstracts, **27**, 2186–2190.

Macesanu, C., 2011, The pseudoacoustic approximation to wave propagation in tti media: SEG Technical Program Expanded Abstracts 2011, 174–178.

Psencik, I., and V. Farra, 2005, First-order ray tracing for qp waves in inhomogeneous, weakly anisotropic media: GEOPHYSICS, **70**, D65–D75.

Sena, A. G., 1991, Seismic traveltime equations for azimuthally anisotropic and isotropic media: Estimation of interval elastic properties: GEOPHYSICS, **56**, 2090–2101.

Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954–1966.